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Dualizing chordal graphs

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Abstract

Although chordal graphs may seem at first to be a poor choice to approach using cycle/cutset graph duality, portions of chordal graph theory can be successfully dualized within the context of nonseparable 3-edge-connected graphs. A simple recognition algorithm for such 'dual-chordal graphs' involves the further restriction to 3-connected (necessarily cubic) graphs. There is also a natural notion of strongly dual-chordal graphs, with a simple description for the planar case.

Keywords: Chordal graph; Duality; Strongly chordal graph

1. Introduction

This paper investigates the class of graphs that is 'dual' to the class of chordal graphs—those graphs whose only induced cycles are triangles—with respect to cycle/cutset duality (a special case of matroidal duality, extending the classical face/vertex duality of plane graphs and polyhedra). See [11] for details of this graph duality and for any graph-theoretic terminology not defined here. (This duality should not be confused with hypergraph duality, which also leads to an interesting notion dual to chordal: the 'dually chordal graphs' of [1].)

It is important to realize that 'duality' will be used only suggestively in the following, as motivation. In particular, one cannot simply get results like Theorem 3 from the corresponding chordal graph result by simply invoking 'duality', since equivalent statements need not 'dualize' into equivalent statements. For instance, every chordal graph has a cutset such that every two edges of that cutset are contained in a triangle—simply take the cutset of edges separating a simplicial vertex from the rest of the graph—but

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$K_{3,3}$ is an example of a dual-chordal graph that has no cycle such that every two edges of that cycle are contained in a size-3 cutset. (See [7,8] for more general discussion of the dangers of trying to dualize concepts instead of specific characterizations; particularly, relevant to Section 3, [8] shows how 'cubic' could be 'dualized' into a broad family of notions.)

Motivated by duality, Section 1 will introduce, motivate, and characterize 'dual-chordal graphs'; in particular, Lemma 4 will reduce their recognition to the cubic graph case. (A different approach is taken in [9].) Section 2 will detail the case of 3-connected graphs, in which dual-chordal graphs have to be cubic. Section 3 will focus on the concept of 'strongly dual-chordal graphs', including a complete classification in the planar 3-connected case.

2. Dual-chordal graphs

A *cycle* of a graph is the edge set of a closed path and a *cutset* of a graph is a minimal set of edges whose deletion would increase the number of connected components. A *chord* of a cycle is an edge that joins two nonconsecutive vertices of the cycle. *Chordal graphs* can be characterized as those graphs in which every cycle of size at least four (so every cycle big enough to have a chord) has a chord. Chordal graphs form one of the most widely studied and seriously applied classes of graphs; see [10]. While there are many characterizations of chordal graphs, few of them seem susceptible to dualization.

There is always a question of what 'graph' should mean when studying graph duality: because size-k cutsets and length-k cycles are duals of each other, allowing *bridges* (size-1 cutsets) or size-2 cutsets would mean also allowing their duals: loops or multiple edges. Hence, the assumption of being 3-*edge-connected* (every cutset having at least three edges) is natural. It will also be useful to focus on *nonseparable graphs* of order at least three (so every two edges are in both a common cycle and a common cutset). For the purposes of the present paper, a *relevant graph* is a nonseparable, 3-edge-connected finite simple graph of order at least three. Therefore, a connected plane graph is relevant if and only if its dual graph (interchanging vertices with faces) is relevant.

A chord e of a cycle C can be characterized as an edge for which C can be partitioned as $P_1 \cup P_2$ such that each $P_i \cup \{e\}$ is a cycle. Thus, in an arbitrary graph G, call an edge e a *cut-chord of a cutset* D if D can be partitioned as $H_1 \cup H_2$ such that each $H_i \cup \{e\}$ is a cutset of G. Equivalently, the cut-chords of D are precisely the bridges in $G - D$. Define a *dual-chordal graph* to be a relevant graph in which every cutset of size at least four has a cut-chord; Fig. 1 shows two examples. The 4-spoked wheel is the smallest relevant graph that is not dual-chordal (because of the size-4 cutset separating the hub from the rim). Observe that a connected plane relevant graph will be dual-chordal if and only if its dual graph is chordal.

The characterization of chordal graphs used above—every cycle of size at least four has a chord—can be restated as follows: a graph is chordal if and only if no induced subgraph is a cycle of length at least four. Theorem 1 can be viewed as dual to

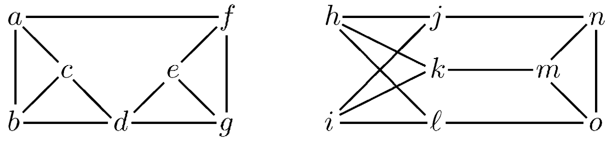

Fig. 1. Two dual-chordal graphs.

this formulation. Toward that goal, observe that a connected graph H is an induced subgraph of a connected graph G if and only if there is a graph G' formed by deleting the edges of a cutset of G, a G'' formed by deleting the edges of a cutset of G', and so on, until the remaining edges form the graph H. Since edge deletion is the dual of edge contraction [11], call a multigraph H a *cycle contraction* of G if there is a graph G' formed by contracting the edges of a cycle of G, a G'' formed by contracting the edges of a cycle of G', and so on, until the remaining edges form the multigraph H. For instance, a graph isomorphic to K_4 is a cycle contraction of the graph on the left in Fig. 1: the cycle $\langle a,b,c,a\rangle$ can first be contracted into vertex c, then the length-2 cycle formed by the double edge cd can be contracted into vertex d (alternatively, the cycle $\langle a,b,d,c,a\rangle$ can be contracted into vertex c, then the loop that came from edge bc can be contracted into vertex c).

Define a k-*linkage* to be the 2-vertex graph that consists of k parallel edges, and notice that these k edges constitute a size-k cutset.

Theorem 1. *A relevant graph is dual-chordal if and only if no cycle contraction is a k-linkage with $k \geqslant 4$.*

Proof. First suppose that G is a dual-chordal graph and that a k-linkage H with $k \geqslant 4$ is a cycle contraction of G (arguing toward a contradiction). Then the k edges of H constitute a cutset D back in G, since H is formed by contracting only cycles that have no edges in common with D. Let e be a cut-chord of D in G, and so a bridge in $G-D$; thus every cycle in G that contains e must contain an edge from D. This means that the edge e will still exist in H, contradicting that H is a k-linkage.

Conversely, suppose G is relevant but is not dual-chordal; suppose in particular, that D is a cutset with $|D| \geqslant 4$ that has no cut-chord. Let H be the cycle contraction resulting from repeatedly contracting cycles that have no edges in common with D. Since $G-D$ has no bridges, H will be a $|D|$-linkage. □

If G is any graph with cutset D, then $G-D$ has two connected components, and so any bridges occurring in $G-D$ can be said to be on the same or on different *sides of the cutset*. (There is no dual notion for cycles, except in the special case of a plane graph.) Define a 3-*cut* to be a size-3 cutset.

Lemma 2. *Suppose G is any relevant graph.*

(1) *A cutset D can have at most $|D|-3$ cut-chords on each side.*
(2) *G is dual-chordal if and only if the cutsets that separate G into 2-edge-connected pieces are precisely the 3-cuts.*

Proof. — To show statement (1), suppose D is a cutset of a relevant graph G and S is one of the sides of D. Let $b \geqslant 0$ denote the number of bridges in S. Consider the tree T with b edges obtained by contracting the nonbridge edges of S, and for each $i \geqslant 0$ let t_i denote the number of degree-i vertices in T. Then $2b = \sum_i it_i$ and $\sum_i t_i = b+1$. (If T consists of a single vertex, then $t_0 = 1$; otherwise, $t_0 = 0$.) Therefore, $\sum_i (3-i)t_i = b+3$, and so $3t_0 + 2t_1 + t_2 \geqslant b+3$. But G being relevant implies that G has no cutsets of size less than three, forcing D to have at least three edges incident with each subgraph of S that corresponds to a single-vertex tree T, at least two edges incident with each subgraph of S that corresponds to a leaf of T, and at least one edge incident with each subgraph of S that corresponds to a degree-2 vertex of T. Hence $|D| \geqslant 3t_0 + 2t_1 + t_2$, which combines with the earlier inequality to show that $b \leqslant |D| - 3$.

Statement (2) follows from the $|D| = 3$ case of statement (1) and the definitions of cut-chord and dual-chordal. □

The extreme value in (1) is achieved by the *Möbius ladder*, formed from a cycle $\langle v_1, v_2, \ldots, v_{2k-4}, v_1 \rangle$, $(k \geqslant 4)$ by inserting the $k-2$ chords ('rungs') $v_i v_{k-2+i}$ for $1 \leqslant i \leqslant k-2$; these chords and two opposite edges of the cycle form a size-k cutset whose deletion leaves $k-3$ bridges on each side. Notice that Möbius ladders with more than three rungs (so $k > 5$) are not dual-chordal.

Statement (2) can be rephrased as follows: *A relevant graph G is dual-chordal if and only if every cutset D big enough to leave a bridge in $G - D$ does leave such a bridge*.

The following theorem corresponds to the result of Jamison [6] that *a graph is chordal if and only if every length-k cycle is the sum of $k-2$ triangles*, where *sum* means the symmetric difference of the edge sets and will be denoted with $\oplus$.

Theorem 3. *A relevant graph is dual-chordal if and only if, for every $k \geqslant 3$, every size-k cutset is the sum of $k-2$ size-3 cutsets.*

Proof. First suppose G is dual-chordal and D is a size-k cutset. Argue by induction on $k \geqslant 3$, with the $k = 3$ case immediate. If $k > 3$, then D has a cut-chord e and $D = H_1 \cup H_2$ such that each H_i has size k_i, $k = k_1 + k_2$, and D is the sum of cutsets $H_1 \cup \{e\}$ and $H_2 \cup \{e\}$. Inductively, each cutset $H_i \cup \{e\}$ is the sum of $(k_i + 1) - 2 = k_i - 1$ size-3 cutsets, and so D is the sum of $k_1 - 1 + k_2 - 1 = k - 2$ size-3 cutsets.

Conversely, suppose D is any size-k cutset of a relevant graph G with $k \geqslant 4$ and $D = D_1 \oplus \cdots \oplus D_{k-2}$ (a sum of $k-2$ size-3 cutsets). Then some D_i must have $|D \cap D_i| > 1$, and so have $|D \cap D_i| = 2$. Let $\{e\} = D_i - D$. Then e is a bridge in $G - D$, and so e is a cut-chord for D. □

Since having edge-connectivity three is a very local property—a 3-cut just has to occur *some*where in a 3-edge-connected graph—Theorem 3 can be viewed as saying that, for relevant graphs, being dual-chordal is a sort of global version of having edge-connectivity three, a sort of uniform statement of edge vulnerability. (This corresponds to relevant chordal graphs being characterized as globally having girth three.)

The following lemma will begin to reduce the recognition problem for dual-chordal graphs to *cubic*—meaning 3-regular—graphs. (Condition (2) or (3) of Theorem 9 can then be combined with this to give a polynomial-time recognition procedure in the 3-connected case.) Suppose a graph G contains a vertex v with $k \geqslant 4$ neighbors $x_1, x_2, \ldots, x_k$. Define G^+ from G by deleting v and inserting any cycle $\langle v_1, v_2, \ldots, v_k, v_1 \rangle$ of new vertices together with new edges $x_i v_i$ whenever $1 \leqslant i \leqslant k$.

Lemma 4. *If G is any relevant graph and if graph G^+ is formed by replacing a vertex v of G with a cycle of length* $\deg v$ *as above, then G is dual-chordal if and only if G^+ is dual-chordal.*

Proof. Suppose G^+ was formed from a relevant graph G by replacing vertex v with a new cycle C as described before the statement of the lemma.

First suppose G is dual-chordal and D^+ is any cutset of G^+ with $|D^+| \geqslant 4$. If C is disjoint from D^+, then D^+ will also be a cutset of G, and any of its cut-chords in G will also be a cut-chord for it in G^+. So suppose $C \cap D^+ \neq \emptyset$. If D^+ contains any two consecutive edges of C incident to a vertex v_i of C, then the edge $x_i v_i$ of G^+ will be a cut-chord of D^+ in G^+. In the remaining case, suppose D^+ has no cut-chord in G^+ (arguing toward a contradiction). Then $G^+ - D^+$ would be bridgeless, as would $G^+ - D^+$ together with the edges of $D^+ - C$ (D^+ being a cutset prevents the additional $\geqslant 2$ edges from being bridges). Thus, $G^+ - (D^+ \cap C)$ would be bridgeless, so every two edges of G^+, one from each side of the cutset D^+, would be in a common cycle in $G^+ - (D^+ \cap C)$. But then, contracting C down to a vertex v to produce G, the cutset of all the edges incident to v in G would have no cut-chord, contradicting G being dual-chordal.

Conversely, if G^+ is dual-chordal, then G will also be dual-chordal by Theorem 1, since it is a cycle contraction of G^+. □

The following shows how different dual-chordal graphs are from chordal graphs.

Theorem 5. *No dual-chordal graph can be a subgraph of a larger dual-chordal graph.*

Proof. Suppose G^- is a subgraph of G^+, both are dual-chordal, $x \in V(G^-)$, and xy is an edge of G^+ not in G^- (arguing toward a contradiction). Since relevant graphs are 3-edge-connected, Theorem 3 implies that xy is in a 3-cut D of G^+. In order for G^- to be 3-edge-connected, none of the edges of D can be in G^-. So none of the side of D that contains y can be in G^-. Suppose for the moment that D contains a second edge incident with x. Then the cutset of G^+ that consists of the $\geqslant 3$ edges incident with x in G^- together with the one edge of D not incident with x would have to have a cut-chord in G^+, contradicting that G^- is 3-edge connected and that the side of D containing y is 2-edge-connected by Lemma 2(2). Therefore, xy is the only edge of D incident with x. Then the cutset of G^+ that consists of the $\geqslant 3$ edges incident with x in G^- together with xy would have to have a cut-chord in G^+, contradicting that G^+ is 3-edge-connected and so that $G^+ - \{xy\}$ is 2-edge-connected. □

Corollary 6. *K_4 is the only graph that is both chordal and dual-chordal.*

Proof. Suppose G is both chordal and dual-chordal; being chordal, G must contain a vertex v whose closed neighborhood $N[v]$ is complete. Since G is relevant, $|N[v]| \geqslant 4$ and so G contains an induced subgraph H that is isomorphic to the dual-chordal graph K_4. By Theorem 5, H must be G itself. □

3. 3-Connected dual-chordal graphs

Imposing a restriction to 3-connected graphs allows a much easier study of dual-chordal graphs. In particular, it is important to note the following.

Lemma 7. *Every 3-connected dual-chordal graph is a cubic graph.*

Proof. Suppose, rather, that a 3-connected dual-chordal graph has a vertex v with degree at least four (arguing toward a contradiction). Let D be the cutset consisting of all the edges incident with v. Then D must have a cut-chord xy with each side of the cutset $\{xy\}$ of $G - v$ containing at least two neighbors of v in G. But then removing both v and x would disconnect the graph, contradicting G being 3-connected. □

It is also convenient that, among 3-connected cubic graphs in general, being a *non-trivial 3-cut*—meaning the edges e_1, e_2, and e_3 of the cutset D are not all incident with the same vertex—is equivalent to the edges of the cutset having six distinct end-points (otherwise, if e_1 and e_2 were incident with v and if $f \neq e_3$ was the third edge incident with v, then e_3 and f would constitute a size-2 cutset, contradicting that G is 3-connected). Therefore, if D is a nontrivial 3-cut of any 3-connected cubic graph, then each side of D must contain a cycle (otherwise, one side of D would be a nontrivial tree, each edge of which would form a size-2 cutset with one of e_1, e_2, or e_3). Having such a cutset is equivalent to a graph being *cyclically 3-edge-connected*. The following corollary is a global version of having 'cyclic edge-connectivity three'.

Corollary 8. *A 3-connected cubic graph is dual-chordal if and only if* each *pair of vertex-disjoint cycles is separated by a 3-cut.*

Proof. First suppose G is a dual-chordal graph and C and C' are two vertex-disjoint cycles in G. Assume D is a minimum-size cutset that separates C from C' with $|D| \geqslant 4$ (arguing toward a contradiction). Then one side of D must contain a bridge e—say the side that contains C; call this side G_C. Let G'_C be the component of $G_C - e$ that contains C. Then there would be a subset $D^- \subset D$ such that $D^- \cup \{e\}$ is a cutset separating C from C'; moreover $|D^-| \leqslant |D| - 2$ (otherwise, the unique edge in $D - D^-$ would form a size-2 cutset with e, contradicting that G is 3-edge-connected). But $|D^- \cup \{e\}| < |D|$ would contradict the assumed minimality of D.

Conversely, suppose G is a 3-connected cubic graph, every pair of vertex-disjoint cycles is separated by a 3-cut, and $D = \{e_1, \ldots, e_k\}$ is any cutset with $k \geqslant 4$. (We show

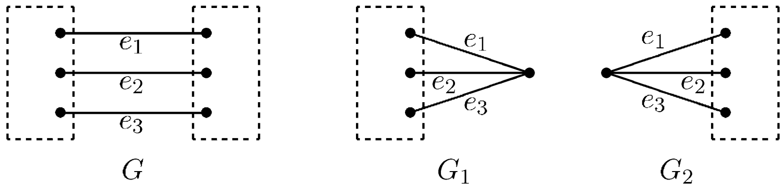

Fig. 2. The decomposition of graph G into graphs G_1 and G_2 along the 3-cut $\{e_1, e_2, e_3\}$, as in Theorem 9(3).

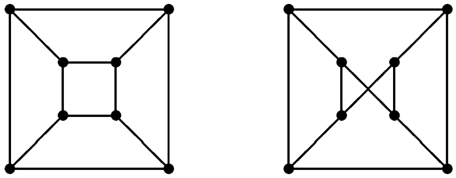

Fig. 3. The cubic, but not dual-chordal, cube and twisted cube (or 4-rung Möbius ladder) graphs mentioned in Theorem 9(4).

that D must have a cut-chord.) If the endpoints of two e_i's are identical, the third edge incident with that vertex will be a cut-chord of D. So assume all the endpoints of the e_is are distinct and let G_1 and G_2 be the two sides of D. Since each G_i is connected, there will be cycles C and C' in G such that $C \cap D = \{e_1, e_2\}$ and $C' \cap D = \{e_3, e_4\}$. If C and C' are vertex disjoint, then they will be separated by a size-3 cutset that will have a unique edge in one G_i; that edge will be a cut-chord for D. If C and C' contain a common vertex in either of G_1 and G_2, then, since they and nonseparable, we can assume C and C' contain common vertices in both; since G is cubic, this means they will contain common edges in each G_i. Then $C \oplus C'$ will contain vertex-disjoint cycles C^* and C'^* such that C^* contains exactly one edge from each of $\{e_1, e_2\}$ and $\{e_3, e_4\}$ and C'^* contains the other two edges. Then, as in the previous case, C^* and C'^* will be separated by a size-3 cutset that contains a cut-chord for D. Thus, in every case, D will have a cut-chord, and so G is dual-chordal. $\square$

In Theorem 9, statement (3) will refer to decomposing a cubic graph G along a nontrivial 3-cut $\{e_1, e_2, e_3\}$ into two smaller cubic graphs G_1 and G_2 as is illustrated in Fig. 2; being $\{K_4, K_{3,3}\}$-*decomposable* means being repeatedly decomposable in that manner into a set of graphs that are each isomorphic to K_4 or $K_{3,3}$. The two graphs mentioned in statement (4) are shown in Fig. 3. The equivalence of statements (3) and (4) is due to Fouquet and Thuillier [5] (their paper [4] contains other notions, such as 'non 3-removable edges', that could be used to give other equivalent statements).

Theorem 9. *The following are equivalent for any* 3-*connected cubic graph* G:

(1) *G is dual-chordal.*
(2) *G reduces to either K_4 or $K_{3,3}$ by repeatedly contracting triangles and induced $K_{2,3}$ subgraphs to vertices.*

(3) *G is $\{K_4, K_{3,3}\}$-decomposable.*
(4) *G contains no subgraph homeomorphic to the cube or the twisted cube.*

Proof. Suppose G is any 3-connected cubic graph. Suppose that G^- results from contacting an induced subgraph H, isomorphic to K_3 or $K_{2,3}$, to a vertex.

Claim. *G is dual-chordal if and only if G^- is dual-chordal.*

If G is dual-chordal, then G^- is a cycle contraction of G (contracting first a 4-cycle and then a 2-cycle of parallel edges when $H \cong K_{2,3}$), and so is dual-chordal by Theorem 1. Conversely, suppose G^- is dual-chordal, and D is any cutset of G with $|D| \geqslant 4$. If D is edge-disjoint from H, then D will also be a cutset of G^-, and any of its cut-chords in G^- will also be a cut-chord for it in G; otherwise, D will contain two edges incident with some vertex x of H, and the third edge incident with w will be a cut-chord for D in G.

For implication (1) $\Rightarrow$ (2), suppose G is dual-chordal and is neither K_4 nor $K_{3,3}$ (the only 3-connected cubic graphs without vertex-disjoint cycles). By Corollary 8, let H be a minimal induced nonseparable subgraph of G that both contains a cycle and is connected to the rest of G by a 3-cut D. Lemma 2(2) shows that the side of D that does not contain H will be 2-edge-connected and so, by repeatedly contracting cycles to vertices, that whole side can eventually be contracted to a vertex. This final contraction G^- of G will not contain two vertex-disjoint cycles (by the minimality of H) and will still be dual-chordal (since Theorem 1 implies that the class of dual-chordal graphs is closed under cycle contraction). Hence G^- is K_4 or $K_{3,3}$, and so H must be K_3 or $K_{2,3}$. Contracting this H and then repeating the above argument will eventually result in a dual-chordal graph isomorphic to K_4 or $K_{3,3}$. Conversely, implication (1) $\Leftarrow$ (2) follows directly from the Claim in the previous paragraph.

For equivalence (2) $\Leftrightarrow$ (3), suppose G is decomposed into G_1 and G_2 as in Fig. 2. Notice that $G_1 \cong K_4$ if and only if G_2 results from contracting a K_3 in G, and $G_1 \cong K_{3,3}$ if and only if G_2 results from contracting a $K_{2,3}$. Hence, if G reduces to K_4 or $K_{3,3}$ by repeated contractions as in (2), then G decomposes into K_4 or $K_{3,3}$ as in (3), each time with either G_1 or G_2 isomorphic to K_3 or $K_{2,3}$. Conversely, any $\{K_4, K_{3,3}\}$-decomposition of G can be thought of as a tree, as in [5], with each edge corresponding to a 3-cut $\{e_1, e_2, e_3\}$ as in Fig. 2 and each leaf corresponding to K_4 or $K_{3,3}$; recursively removing leaves from this tree corresponds to reducing G to K_4 or $K_{3,3}$ by repeated contractions as in (2).

Equivalence (3) $\Leftrightarrow$ (4) is Theorem 4.4 of [5]. $\square$

Statement (2) of Theorem 9 can, of course, be equivalently stated in terms of G reducing to a 3-linkage by repeatedly contracting triangles and $K_{2,3}$ subgraphs (for instance, contract the $K_{2,3}$ induced by $\{h, i, j, k, \ell\}$ and the triangle induced by $\{m, n, o\}$ in the graph on the right in Fig. 1). Since triangles and copies of $K_{2,3}$ can be found in polynomial time, G can be reduced in polynomial time to a graph without such subgraphs, and G will be dual-chordal if and only if a 3-linkage results. By Fouquet and Thuillier [5], the number of each type of contraction is independent of the order and choice of contractions, and so the number of copies of $K_{3,3}$ in the decomposition in Theorem 9(3) is uniquely determined. Since this number equals the *crossing number* of G—the fewest number of edges crossings required in a plane embedding of G—the crossing number of every 3-connected cubic dual-chordal graph can be found in polynomial time. Thus, using Lemma 4 to produce a cubic graph with the same crossing number, *the crossing number of every 3-connected dual-chordal graph can be found in polynomial time*. This parallels the valuable phenomenon of many NP-complete problems becoming polynomial when restricted to chordal graphs.

4. Strongly dual-chordal graphs

A chord e of a cycle C is a *strong chord* if C can be partitioned as $P_1 \cup P_2$ such that at least one $P_i \cup \{e\}$ is a cycle of even length. (By this definition, every chord of an odd-length cycle is a strong chord.) A *strongly chordal graph* [3] is a chordal graph with the additional property that every cycle of even length at least six (or, equivalently, every cycle of length at least five) has a strong chord. While there are many other characterizations of strongly chordal graphs [10], few of them seem susceptible to dualization.

Call an edge e of a graph G a *strong cut-chord* of a cutset D if D can be partitioned as $H_1 \cup H_2$ such that at least one $H_i \cup \{e\}$ is an even-size cutset of G (which is trivially true when $|D|$ is odd). Define a *strongly dual-chordal graph* to be a relevant dual-chordal graph in which every cutset of even size at least six (or, equivalently, every cutset of size at least five) has a strong cut-chord. The dual-chordal graph on the left in Fig. 4 is not strongly dual-chordal, as shown by the cutset $\{ad, de, fh, fi, gh, gi\}$; the graph on the right is strongly dual-chordal with, for instance, pq a strong cut-chord for the cutset $\{k\ell, km, mn, no, op, \ell s\}$. Of the seven cubic dual-chordal graphs of order 10 (out of 14 relevant cubic graphs of order 10) only the graphs in Fig. 5 and the leftmost graph in Fig. 9 are strongly-dual-chordal. Observe that a connected plane relevant graph will be strongly dual-chordal if and only if its dual graph is strongly chordal.

Theorem 10 below corresponds to the result of Dahlhaus et al. [2] that *a chordal graph is strongly chordal if and only if every cycle with length at least five has two chords that form a triangle with an edge of the cycle*.

Theorem 10. *A dual-chordal graph is strongly dual-chordal if and only if every cutset D with size at least five has two cut-chords that form a size-3 cutset with an edge of D.*

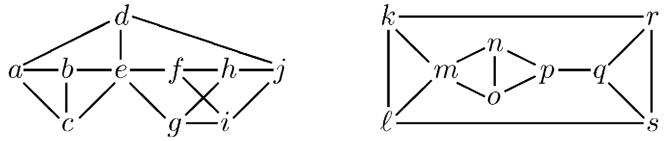

Fig. 4. Two dual-chordal graphs, but only the one on the right is strongly dual-chordal.

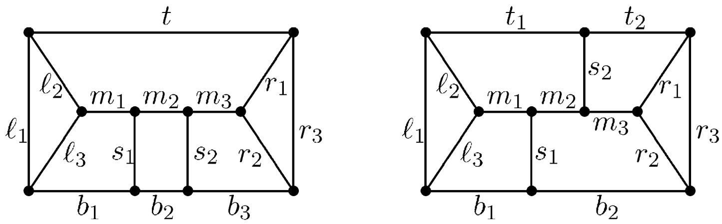

Fig. 5. Two 3-connected strongly-chordal graphs drawn as triangular prisms with parallel struts s_1 and s_2.

Proof. First suppose G is strongly dual-chordal and D is any cutset of size at least five. Repeating the use of the definition of strong cut-chord on $H_i \cup \{e\}$, there is a size-4 cutset D' that consists of three edges of D and one cut-chord e of D. Since G is dual-chordal, D' partitions as $H_1' \cup H_2'$ with a cut-chord f of D' such that one $H_i' \cup \{f\}$ will consist of one edge $d \in D$ plus e and f. Since f is also a cut-chord of D, $\{d, e, f\}$ is the desired size-3 cutset.

Conversely, suppose G is dual-chordal, each cutset of size at least five has a size-3 cutset consisting of one edge in the cutset and two cut-chords, and D is a particular cutset with size at least five. Let $\{d, e, f\}$ be the size-3 cutset with $d \in D$ and with e and f cut-chords of D. By the definition of cut-chord, D partitions as $H_1 \cup H_2$ where each $H_i \cup \{e\}$ is a cutset, and also as $H_1' \cup H_2'$ where each $H_i' \cup \{f\}$ is a cutset. Without loss of generality, say that $d \in H_1 \cap H_1'$. Then $D'' = (H_1 \cup \{e\}) \oplus (H_2' \cup \{f\})$ is in the cutset space [11] of G with $\{d, e, f\} \subseteq D''$ and $D'' - \{d, e, f\} \subseteq D$. Since the latter containment is proper and D is a cutset, $D'' - \{d, e, f\} = \emptyset = (H_1 - \{d\}) \oplus H_2'$, and so $|H_1 - \{d\}| = |H_2'|$. Thus precisely one of $|H_1|$ and $|H_2'|$ must be odd. If $|D|$ is even, then either $|H_1|$ is odd, making both cutsets $H_i \cup \{e\}$ even and so e a strong cut-chord of D, or $|H_2'|$ is odd, making both cutsets $H_i' \cup \{f\}$ even and so f a strong cut-chord of D. Finally, if $|D|$ is odd, then both e and f are automatically strong cut-chords of D. □

The $\{K_4, K_{3,3}\}$-decomposition in Theorem 9 of any 3-connected dual-chordal graph G—which must be cubic by Lemma 7—imparts a tree structure as follows: if G is decomposed into G_1 and G_2 as in Fig. 2, form the 2-vertex tree whose vertices are the edge sets of G_1 and G_2 and whose edge corresponds to the nontrivial 3-cut $\{e_1, e_2, e_3\}$. Repeat this recursively for each of G_1 and G_2 until no vertex of the tree (no subgraph G_i) can be further decomposed. Each vertex of the final tree will correspond to a K_4 or $K_{3,3}$ subgraph of G, with the edge between two vertices of the tree corresponding to the intersection of the edge sets of the two corresponding K_4 or $K_{3,3}$ subgraphs of G. Indeed, the edges of the tree will correspond to all the nontrivial 3-cuts of G. For instance, the graph on the left in Fig. 5 has tree (indeed, path) structure

$$\ell_1\ell_2\ell_3 t m_1 b_1 \equiv t m_1 m_2 s_1 b_1 b_2 \equiv t m_2 m_3 s_2 b_2 b_3 \equiv t m_3 b_3 r_1 r_2 r_3,$$

where the vertex $\ell_1\ell_2\ell_3 t m_1 b_1$ abbreviates $\{\ell_1, \ell_2, \ell_3, t, m_1, b_1\}$ and corresponds to a K_4, and the leftmost edge corresponds to the 3-cut $\{t, m_1, b_1\}$ (the intersection of the two

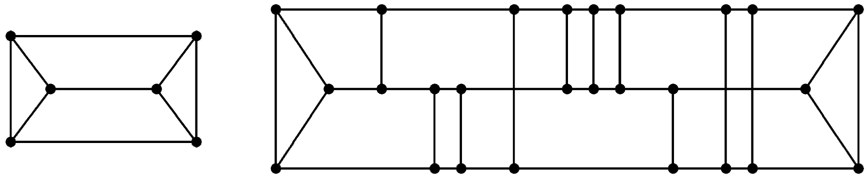

Fig. 6. Triangular prisms with zero and ten parallel struts.

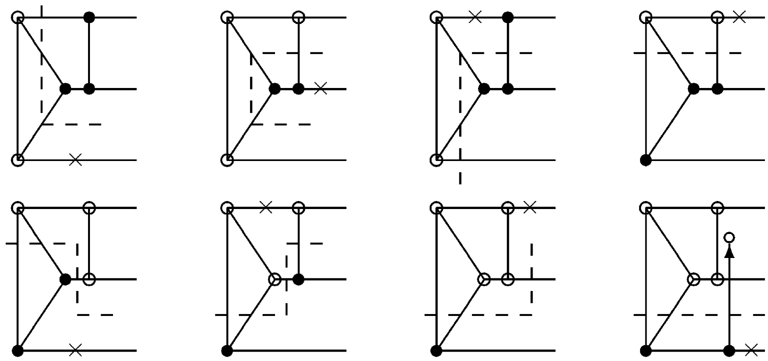

Fig. 7. Parts of cutsets of size at least six that have strong cut-chords.

vertices on the left); the graph on the right has tree (indeed, path) structure

$$\ell_1\ell_2\ell_3t_1m_1b_1 \equiv t_1m_1m_2s_1b_1b_2 \equiv t_1t_2s_2m_2m_3b_2 \equiv t_2m_3b_2r_1r_2r_3.$$

This tree structure is particularly simple when G is both planar and strongly dual-chordal: the proof of Theorem 11 will show that such trees must be paths and that G can be embedded as a triangular prism with inserted *parallel struts*, as illustrated in Figs. 5 and 6.

Theorem 11. *The planar* 3*-connected strongly dual-chordal graphs are precisely* K_4 *and triangular prisms with any number of parallel struts* (*as illustrated in Figs.* 5 *and* 6).

Proof. Both K_4 and the triangular prism are each strongly dual-chordal. Suppose G is a triangular prism with at least one parallel strut, embedded as in Fig. 6 (toward showing that G is strongly dual-chordal). Consider the left triangle and notice that (redrawing the graph if necessary) the closest strut to it can be assumed to be between the upper two horizontal edges as shown in the examples in Fig. 7. Suppose D is any k-cut with k even and $k \geqslant 6$ and argue by induction. If D contains no edge of the left triangle, then there is a strong cut-chord by the inductive hypothesis (after contracting the triangle). Otherwise, D will be as illustrated in Fig. 7, indicated both by the dashed line and as separating the 'solid' and 'hollow' vertices. The left three examples in the top row show the cases in which D contains both slanted edges of the

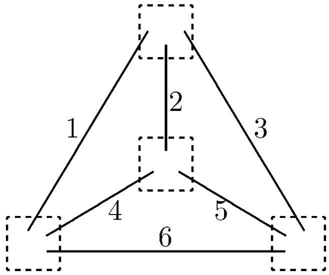

Fig. 8. Form of G so as to have a vertex with degree greater than two in T.

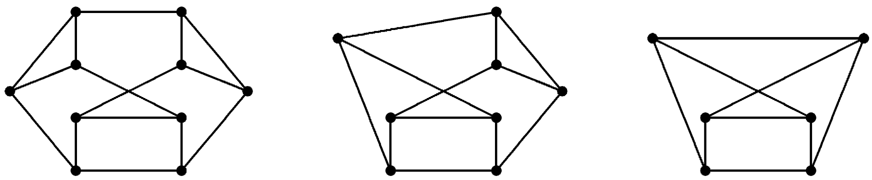

Fig. 9. Three nonplanar 3-connected strongly dual-chordal graphs.

triangle; the upper-right and lower-left examples show the cases in which D contains the vertical and downward sloped edge of the triangle; the three remaining show the cases in which D contains the vertical and upward sloped edge of the triangle (in the lower-right case, the arrow indicates the closest strut to the triangle that has a solid vertex on the bottom horizontal edge and a hollow vertex on either the middle or top edge). A strong cut-chord for D is shown, marked with $\times$, in each case.

Conversely, suppose G is a planar 3-connected, strongly dual-chordal (cubic) graph with a tree structure T determined as in Theorem 9 (toward first showing that T is a path). Suppose T has a vertex V of degree greater than two (arguing toward a contradiction). By Theorem 9(2), G must be of the form shown in Fig. 8, where each dotted box represents an induced *block* (2-connected subgraph) of G. Indeed, since V has degree greater than two, the edges of T incident with V must each correspond to a 3-cut of G such that the edges of each 3-cut must have distinct endpoints in G. Hence, at least three of the blocks in Fig. 8 must be *nontrivial* in that the endpoints of the three numbered edges ending there must be three distinct vertices of G.

Suppose, say, at least the three 'outside' blocks are nontrivial. Since G is cubic, there is a cutset D that consists of the two edges in each of the outside blocks that are incident with the endpoints of edges 2, 4, and 5. Then edges 2, 4, and 5 are the only cut-chords of D, but none is a strong cut-chord (contradicting that G is strongly dual-chordal). Thus T must be a path, and an induction shows that G must be either K_4 or a triangular prism with parallel struts. □

Although the nonplanar 3-connected strongly dual-chordal graphs seem harder to analyze, they may be very limited in number; it is unknown whether they all have one of the three graphs in Fig. 9 as a cycle contraction.

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